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A Characterization of Strict Local Minimizers of Order One for Nonsmooth Static Minmax Problems

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We present first-order necessary optimality conditions for a nonsmooth static minmax problem with inequality constraints. These conditions, after some modification, turn out to characterize strict local minimizers of order one for the given problem.

1. INTRODUCTION

The notion of a strict local minimizer of order m (for $m = 1$, called also a strongly unique minimizer) plays an important role in the convergence analysis of iterative numerical methods (see [1]) and in stability results (see, e.g., [2, 3]). An important feature of such minimizers is that, in the presence of constraint qualifications, they can be completely characterized, which means that there is no gap between the necessary and sufficient conditions. Corresponding results for standard nonlinear programming problems with both inequality and equality constraints are stated, for the cases where $m = 1$ or 2, in [4–6].

The aim of this paper is to extend the characterization obtained for $m = 1$ in [6, Theorem 5] to a certain class of nonsmooth static minmax

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problems with inequality constraints. We consider problems in which the objective function satisfies the assumptions of Clarke's theorem on "max functions" [7, Theorem 2.1], while the functions defining inequality constraints are locally Lipschitzian and regular in Clarke's sense. Section 2 is devoted to necessary optimality conditions which are satisfied by all local minimizers (not necessarily strict) for the given problem. These conditions include a restriction on the number of nonzero multipliers which is well known in the differentiable case (see, e.g., [8, Theorem 1]). In Section 3, we show that the previous necessary conditions can be modified so as to obtain a characterization of strict local minimizers of order one.

We assume that the reader is familiar with the basic calculus of Clarke's generalized gradients (see [9, Chap. 2]). We use the notation $\partial f(x)$ for the generalized gradient of a locally Lipschitzian function $f : \mathbb{R}^n \to \mathbb{R}$ at x. Let us remember that f is called *regular* (or *subdifferentially regular*) at x if the usual one-sided directional derivative $f'(x; d)$ exists for all d and is equal to the generalized directional derivative $f^\circ(x; d)$.

Consider the nonlinear programming problem

$$\min\{f(x) \mid x \in S\}, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and S is a nonempty subset of $\mathbb{R}^n$ defined by

$$S := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \forall\, i \in I\} \tag{2}$$

with $I := \{1, \ldots, p\}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in I)$. A particular case of problem (1), (2) is the *static minmax problem* in which the objective function f is given by

$$f(x) := \sup_{y \in Y} \phi(x, y), \tag{3}$$

where $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and Y is a nonempty subset of $\mathbb{R}^m$. We assume that the value $f(x)$ given by (3) is finite for all $x \in \mathbb{R}^n$.

For $x_0 \in \mathbb{R}^n$ and $\delta > 0$, we denote $B(x_0, \delta) := \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq \delta\}$. We say that $x_0 \in S$ is a *local minimizer* for problem (1) if there exists $\varepsilon > 0$ such that

$$f(x) \geq f(x_0) \text{ for all } x \in S \cap B(x_0, \varepsilon).$$

Let $m \geq 1$ be an integer. We say that $x_0 \in S$ is a *strict local minimizer of order m* for problem (1) if there exist $\varepsilon > 0$, $\beta > 0$ such that

$$f(x) \geq f(x_0) + \beta\|x - x_0\|^m \text{ for all } x \in S \cap B(x_0, \varepsilon).$$

Throughout the paper, we will use the following notation for a given $x \in \mathbb{R}^n$:

$$I(x) := \{i \in I \mid g_i(x) = 0\},$$
$$M(x) := \{y \in Y \mid \phi(x, y) = f(x)\}.$$

2. FIRST-ORDER NECESSARY OPTIMALITY CONDITIONS

It is known (see remarks in [10; 11, p. 273]) that Clarke's theorem on "max functions" can be used to derive first-order necessary optimality conditions for the static minmax problem (1)–(3) in which the functions ϕ and g_i are possibly nondifferentiable. However, a simple repetition of the argument presented in [10] yields a result in which the restriction of the overall number of Lagrange multipliers to $n+1$ is not present. Therefore, in this section we present a detailed proof of first-order necessary optimality conditions for a nondifferentiable problem (1)–(3) with appropriate modifications which allow us to obtain the restriction.

Let $x_0 \in S$. Consider the unconstrained optimization problem

$$\min\{h(x) \mid x \in \mathbb{R}^n\}, \tag{4}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$h(x) := \max\{f(x) - f(x_0), g_i(x) \mid i \in I\}. \tag{5}$$

Observe that $h(x_0) = 0$. The relationship between problems (1), (2) and (4), (5) is given in the following lemma.

LEMMA 1. *If* x_0 *is a local minimizer for* (1), (2), *then* x_0 *is also a local minimizer for* (4), (5).

Proof. The proof is elementary (see, e.g., [12, Theorem 2.1]). ∎

The following lemma will be used in the proof of Theorem 4 below. The symbol $\operatorname{co} A$ will denote the convex hull of A.

LEMMA 2. *For any subsets* A, B *of* $\mathbb{R}^n$, *we have*

$$\operatorname{co}((\operatorname{co} A) \cup B) = \operatorname{co}(A \cup B).$$

Proof. By the definition of a convex hull, we have $\operatorname{co} A \subset \operatorname{co}(A \cup B)$ and $B \subset \operatorname{co}(A \cup B)$. Since $\operatorname{co}(A \cup B)$ is convex, this implies $\operatorname{co}((\operatorname{co} A) \cup B) \subset \operatorname{co}(A \cup B)$. The opposite inclusion is obvious. ∎

We will need some additional assumptions concerning problem (1)–(3).

Condition 3. Assume that: (a) the set Y is compact;

(b) $\phi(x, y)$ is upper semicontinuous in (x, y);

(c) ϕ is locally Lipschitzian in x, uniformly for y in Y (see [11, p. 224] for a precise statement);

(d) ϕ is regular in x—that is, $\phi_x^\circ(x, y; \cdot) = \phi_x'(x, y; \cdot)$, the derivatives being with respect to x;

(e) the set-valued map $\partial_x \phi(x, y)$ is upper semicontinuous in (x, y);

(f) g_i, $i \in I$, are locally Lipschitzian and regular at x_0.

Clarke [7, Theorem 2.1] has shown that, under Conditions (a)–(e), the maximum function f defined by (3) is locally Lipschitzian; $f'(x;d)$ exists and is given by the formula

$$f'(x;d) = f^{\circ}(x;d) = \max\{\zeta \cdot d \mid \zeta \in \partial_x \phi(x,y),\, y \in M(x)\}, \tag{6}$$

where $\zeta \cdot d$ denotes the inner product of vectors ζ and d. Moreover, the "sup" in (3) can be replaced by "max," and the generalized gradient $\partial f(x)$ is given by

$$\partial f(x) = \mathrm{co} \bigcup_{y \in M(x)} \partial_x \phi(x,y). \tag{7}$$

The following result is a generalization of [8, Theorem 1].

THEOREM 4. *Let x_0 be a local minimizer for* (1)–(3), *and suppose that Condition* 3 *holds. Then there exist a positive integer q and vectors $y_i \in M(x_0)$ together with scalars $\lambda_i \geq 0$, $i = 1, \dots, q$, $\mu_j \geq 0$, $j \in I$, such that*

$$0 \in \sum_{i=1}^{q} \lambda_i \partial_x \phi(x_0, y_i) + \sum_{j=1}^{p} \mu_j \partial g_j(x_0), \tag{8}$$

$$\mu_j g_j(x_0) = 0, \quad j \in I. \tag{9}$$

Furthermore, if α is the number of nonzero λ_i, and β is the number of nonzero μ_j, then

$$1 \leq \alpha + \beta \leq n + 1. \tag{10}$$

Proof. Let x_0 be a local minimizer for (1)–(3); then x_0 is a local minimizer for (4)–(5) by Lemma 1. Define $g_0(x) := f(x) - f(x_0)$ and $I_0(x_0) := \{0\} \cup I(x_0)$. We have $h(x) = \max\{g_i(x) \mid i \in I_0(x_0)\}$ and $h(x_0) = 0 = g_0(x_0)$.

By Condition 3 and [7, Theorem 2.1], f is locally Lipschitzian and regular at x_0, so that g_0 has the same properties. Then, using [9, Propositions 2.3.2 and 2.3.12], we obtain

$$0 \in \partial h(x_0) = \mathrm{co} \bigcup_{j \in I_0(x_0)} \partial g_j(x_0) = \mathrm{co}\left\{\partial f(x_0) \cup \bigcup_{j \in I(x_0)} \partial g_j(x_0)\right\}.$$

Now, applying formula (7) to $\partial f(x_0)$, we deduce

$$0 \in \mathrm{co}\left\{\mathrm{co}\left(\bigcup_{y \in M(x_0)} \partial_x \phi(x_0, y)\right) \cup \bigcup_{j \in I(x_0)} \partial g_j(x_0)\right\}.$$

By Lemma 2, this is equivalent to

$$0 \in \mathrm{co}\left\{ \bigcup_{y\in M(x_0)} \partial_x \phi(x_0, y) \cup \bigcup_{j\in I(x_0)} \partial g_j(x_0) \right\}.$$

Hence, by Caratheodory's theorem, there exist scalars $\lambda_i > 0, i = 1, \ldots, \alpha$, $\eta_l > 0, l = 1, \ldots, r$, such that

$$1 \le \alpha + r \le n + 1 \tag{11}$$

and

$$0 = \sum_{i=1}^{\alpha} \lambda_i u_i + \sum_{l=1}^{r} \eta_l w_l \quad \text{for some } u_i \in \bigcup_{y\in M(x_0)} \partial_x \phi(x_0, y),$$
$$w_l \in \bigcup_{j\in I(x_0)} \partial g_j(x_0). \tag{12}$$

(Here, we allow the case where if $\alpha = 0$, then the set of multipliers λ_i is empty; similarly, if $r = 0$, then there are no multipliers η_l.)

For each $j \in I(x_0)$, we define

$$J(j) := \left\{ l \in \{1, \ldots, r\} \mid w_l \in \partial g_j(x_0) \Big\backslash \bigcup_{\substack{s\in I(x_0)\\ s<j}} \partial g_s(x_0) \right\}$$

and

$$v_j := \begin{cases} (\sum_{l\in J(j)} \eta_l w_l)/(\sum_{l\in J(j)} \eta_l), & \text{if } J(j) \neq \emptyset, \\ \text{an arbitrary element of } \partial g_j(x_0), & \text{if } J(j) = \emptyset, \end{cases}$$

$$\mu_j := \begin{cases} \sum_{l\in J(j)} \eta_l, & \text{if } J(j) \neq \emptyset, \\ 0, & \text{if } J(j) = \emptyset. \end{cases}$$

Then $\mu_j \ge 0$ and $v_j \in \partial g_j(x_0)$ (by the convexity of $\partial g_j(x_0)$) for all $j \in I(x_0)$. Moreover, Condition (12) implies that

$$0 = \sum_{i=1}^{\alpha} \lambda_i u_i + \sum_{j\in I(x_0)} \mu_j v_j,$$

where $u_i \in \partial_x \phi(x_0, y_i)$ for some $y_i \in M(x_0)$, $i = 1, \ldots, \alpha$. To obtain (8) and (9), let $\mu_j = 0$ for $j \in I\backslash I(x_0)$. Also, if $\alpha > 0$, then we can take $q = \alpha$, and all λ_i are nonzero. However, if $\alpha = 0$, then let $q = 1$, $\lambda_1 = 0$, and y_1 be an arbitrary element of $M(x_0)$.

Let β be the number of nonzero μ_j. Then it follows from our construction that β is not greater than the number r of all η_l. Moreover, if $r > 0$, then $\beta > 0$. Hence, Condition (11) implies (10). ∎

Remark 5. First-order necessary optimality conditions for nonsmooth minmax problems have been studied in [13]. However, the conditions obtained there are in rather abstract form; in particular, no representation of the constraint set as a system of inequalities is considered. We hope that our conditions given in Theorem 4 can be applied more easily to nonsmooth problems in which inequality constraints are given explicitly. A simple example of such a problem is given at the end of this paper.

3. CHARACTERIZATIONS OF STRICT LOCAL MINIMIZERS OF ORDER ONE

We begin by reviewing some result for the standard nonlinear programming problems (1) and (2). Throughout this section, we assume that the following constraint qualification is satisfied at x_0:

Condition 6. For each $\eta \in \mathbb{R}^p$ satisfying the conditions

$$\eta_i = 0, \quad \forall\, i \in I \backslash I(x_0); \qquad \eta_i \geq 0, \quad \forall\, i \in I(x_0),$$

the following implication holds:

$$z_i^* \in \partial g_i(x_0)\ (\forall\, i \in I), \quad \sum_{i=1}^{p} \eta_i z_i^* = 0 \quad \Rightarrow \quad \eta = 0.$$

LEMMA 7. *Consider problem* (1), (2), *where the functions f and g_i, $i \in I$, are locally Lipschitzian and possess one-sided directional derivatives at $x_0 \in S$. Suppose that Condition* 6 *holds. Then x_0 is a strict local minimizer of order one for* (1), (2) *if and only if*

$$f'(x_0; d) > 0, \quad \forall\, d \in C(x_0) \backslash \{0\},$$

where

$$C(x_0) := \{d \in \mathbb{R}^n \mid g_i'(x_0; d) \leq 0, \forall\, i \in I(x_0)\}. \tag{13}$$

Proof. This result follows from [6, Theorem 4] and the well-known equality $f^K(x_0; \cdot) = f'(x_0; \cdot)$, which holds for a locally Lipschitzian f, directionally differentiable at x_0 (here f^K denotes the *contingent epiderivative* of f; see [14, Sect. 6.1]). ∎

We now formulate the main result of this section.

THEOREM 8. *Consider problems* (1)–(3). *Suppose that Conditions* 3 *and* 6 *are satisfied. Then x_0 is a strict local minimizer of order one for* (1)–(3) *if* *and only if the following conditions hold*:

(a) $C(x_0) \cap \{d \in \mathbb{R}^n \backslash \{0\} \mid \max\{\zeta \cdot d \mid \zeta \in \partial_x \phi(x_0, y), y \in M(x_0)\} = 0\} = \emptyset$;

(b) *there exist a positive integer* α *and vectors* $y_i \in M(x_0)$, *together with scalars* $\lambda_i > 0, i = 1, \ldots, \alpha, \mu_j \geq 0, j \in I$, *such that*

$$0 \in \sum_{i=1}^{\alpha} \lambda_i \partial_x \phi(x_0, y_i) + \sum_{j=1}^{p} \mu_j \partial g_j(x_0), \tag{14}$$

$$\mu_j g_j(x_0) = 0, \quad j \in I, \tag{15}$$

$$1 \leq \alpha + \beta \leq n+1, \tag{16}$$

where β *is the number of nonzero* μ_j.

Proof. (i) Necessity: Suppose that x_0 is a strict local minimizer of order one for problem (1)–(3). Then it is also a local minimizer for (1)–(3); therefore, Theorem 4 implies that there exist a positive integer q and vectors $y_i \in M(x_0)$ together with scalars $\lambda_i \geq 0, i = 1, \ldots, q, \mu_j \geq 0, j \in I$, such that Conditions (8)–(10) hold. Suppose that all λ_i are zero; then Condition (8) takes on the form

$$0 \in \sum_{j=1}^{p} \mu_j \partial g_j(x_0).$$

Then it follows from Condition 6 and equalities (9) that all μ_j are zero, a contradiction with the left-hand inequality in (10). Therefore, at least one λ_i must be nonzero, which means that Condition (b) holds. Condition (a) follows from Lemma 7 and formula (6).

(ii) Sufficiency: By Lemma 7 and formula (6), it suffices to show that

$$f'(x_0; d) = \max\big\{\zeta \cdot d \mid \zeta \in \partial_x \phi(x_0, y), y \in M(x_0)\big\} > 0 \tag{17}$$

for all $d \in C(x_0) \backslash \{0\}$. Define $\nu_i := \lambda_i / \lambda, i = 1, \ldots, \alpha$, where $\lambda := \sum_{i=1}^{\alpha} \lambda_i > 0$. Then, using Condition (14) and the equality $\sum_{i=1}^{\alpha} \nu_i = 1$, we deduce

$$\begin{aligned} 0 &\in \lambda \sum_{i=1}^{\alpha} \nu_i \partial_x \phi(x_0, y_i) + \sum_{j=1}^{p} \mu_j \partial g_j(x_0) \\ &\in \lambda \operatorname{co} \bigcup_{y \in M(x_0)} \partial_x \phi(x_0, y) + \sum_{j=1}^{p} \mu_j \partial g_j(x_0) \\ &= \lambda \partial f(x_0) + \sum_{j=1}^{p} \mu_j \partial g_j(x_0), \end{aligned}$$

where the last equality follows from (7). By (15), we have $\mu_j = 0$ for all $j \in I \backslash I(x_0)$. Therefore,

$$0 \in \lambda \partial f(x_0) + \sum_{j \in I(x_0)} \mu_j \partial g_j(x_0).$$

Since f and g_j, $j \in I$, are regular at x_0 by formula (6) and Condition 3(f), we apply [9, Corollary 3, p. 40] to obtain

$$0 \in \partial\left(\lambda f + \sum_{j \in I(x_0)} \mu_j \partial g_j\right)(x_0). \tag{18}$$

Now, take any $d \in C(x_0)\backslash\{0\}$. Our assumptions on f and g_j imply that the function $k := \lambda f + \sum_{j \in I(x_0)} \mu_j g_j$ is regular at x_0 (see [9, Proposition 2.3.6(c)]) Hence, from (18) and the definition of generalized gradient, it follows that

$$\begin{aligned} 0 \le k^\circ(x_0;d) = k'(x_0;d) = \lambda f'(x_0;d) + \sum_{j \in I(x_0)} \mu_j g_j'(x_0;d) \\ = \lambda \max\{\zeta \cdot d \mid \zeta \in \partial_x \phi(x_0, y), y \in M(x_0)\} + \sum_{j \in I(x_0)} \mu_j g_j'(x_0;d) \\ \le \lambda \max\{\zeta \cdot d \mid \zeta \in \partial_x \phi(x_0, y), y \in M(x_0)\}, \end{aligned} \tag{19}$$

where the last inequality is a consequence of (13). Now, the desired inequality (17) follows from (19) and Condition (a) of the theorem. ∎

EXAMPLE 9. Consider problem (1)–(3), where $n = m = p = 1$, $Y = [0, 1]$, and the functions ϕ and g_1 are given, respectively, by

$$\phi(x, y) := \begin{cases} x + 2y, & \text{if } y \ge x, \\ 2x + y, & \text{if } y < x, \end{cases}$$

and $g_1(x) := |x - \frac{3}{2}| - \frac{1}{2}$. Then $S = [1, 2]$, and it can be seen directly that

$$f(x) = \max_{y \in Y} \phi(x, y) = \begin{cases} x + 2, & \text{if } x \le 1, \\ 2x + 1, & \text{if } x > 1. \end{cases}$$

This implies that the only optimal minmax solution is $x_0 = 1$.

We shall show that the same result can be obtained by applying Theorems 4 and 8, without finding a direct formula for f. First, we apply Theorem 4 to an arbitrary point $x_0 \in S$. For any such x_0, we have $M(x_0) = \{1\}$; hence we may assume $q = 1$ in (8). If $x_0 \in (0, 2)$, then $g_1(x_0) < 0$, and so $\mu_1 = 0$ by (9). Since

$$\partial_x \phi(x, y) = \begin{cases} \{1\}, & \text{if } x < y, \\ [1, 2], & \text{if } x = y, \\ \{2\}, & \text{if } x > y, \end{cases}$$

Condition (8) cannot hold with $\lambda_1 > 0$. Similarly, if $x_0 = 2$, then (8) reduces to $0 = 2\lambda_1 + \mu_1$, which cannot be satisfied with $\lambda_1 \geq 0$, $\mu_1 \geq 0$, $\lambda_1 + \mu_1 > 0$. The only candidate for an optimal solution is $x_0 = 1$, for which (8) has the form $0 \in \lambda_1[1, 2] - \mu_1$; all Conditions (8)–(10) are then satisfied with $\lambda_1 = \mu_1 = 1$.

We now apply Theorem 8 to prove that $x_0 = 1$ is a strict local minimizer of order one for (1)–(3). Since $0 \notin \partial g_1(x_0) = \{-1\}$, Condition 6 is satisfied. Then, it is sufficient to verify Condition (a) of the theorem, which follows from the equalities $C(x_0) = \{d \in \mathbb{R} \mid d \geq 0\}$ and, for all $d \geq 0$,

$$\begin{aligned} \max\{\zeta \cdot d \mid \zeta \in \partial_x \phi(x_0, y), y \in M(x_0)\} &= \max\{\zeta \cdot d \mid \zeta \in \partial_x \phi(1, 1)\} \\ &= \max\{\zeta \cdot d \mid \zeta \in [1, 2]\} = 2d. \end{aligned}$$

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